# Outer limits of subdifferentials for min-max type functions

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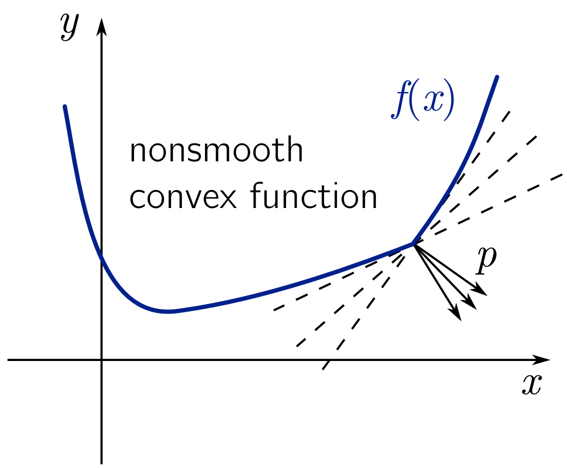
# Simple subdifferentials

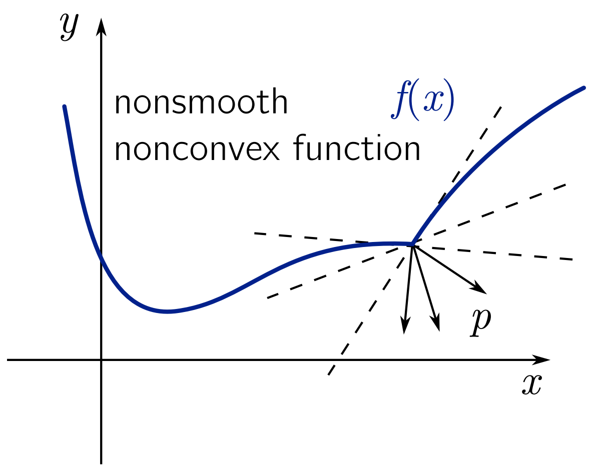
Fréchet subdifferential $\partial f$ is a generalisation of the subdifferential to nonconvex functions:

$$\partial f(x) = \left\{ v \,|\, \liminf_{x' \to x} \frac{f(x') - f(x) - \langle v, x' - x \rangle}{\|x - x'\|} \geq 0 \right\}.$$

Compare with the subdifferential of a convex function:

$$\partial f(x) = \{ v \,|\, f(x') \geq f(x) + \langle v, x' - x \rangle \quad \forall x' \in \mathbb{R}^n \}.$$



# An old result

Let $\varphi_1, \varphi_2, \ldots, \varphi_r\colon \mathbb{R}^n \to \mathbb{R}$ be lower semicontinuous and approximate convex at $\bar{x}$. Then for the limiting subdifferential

$$\bar{\partial}\left(\min_{i\in 1:r} \varphi_i\right)(\bar{x}) \subset \left(\bigcap_{i\in I(\bar{x})} \partial\varphi_i(\bar{x})\right) \cup \left(\bigcup_{g\in\mathcal{S}^{n-1}} C_{\bar{x}}(g)\right),$$

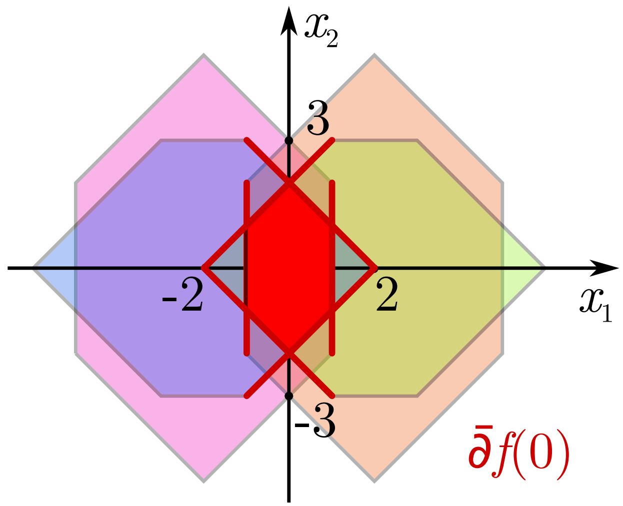
$$C_{\bar{x}}(g) = \left\{\bar{v} \,|\, \bar{v} \in \operatorname*{Arg\,max}_{\partial\varphi_{i_0}(\bar{x})}\langle v, g\rangle,\ \langle \bar{v}, g\rangle = \min_{i\in I(\bar{x})} \max_{v\in\partial\varphi_i(\bar{x})} \langle v, g\rangle\right\}$$

'Approximate convex' means that for every $\varepsilon > 0$ there is $\delta > 0$ such that for $x, y \in \bar{x} + \delta\mathcal{B}$ and $\lambda \in (0, 1)$

$$\varphi(x + \lambda(y - x)) \leq \varphi(x) + \lambda(\varphi(y) - \varphi(x)) + \varepsilon\lambda(1 - \lambda)\|x - y\|.$$

[Ngai, Luc, Thera (2000)], [Roshchina (2010)], also see [Daniilidis et al 2004, 2009]

# An old result




Let $\varphi_1, \varphi_2, \ldots, \varphi_r \colon \mathbb{R}^n \to \mathbb{R}$ be lower semicontinuous and approximate convex at $\bar{x}$. Then

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# Motivation

Error bound bound [Fabian, Henrion, Kruger and Outrata, 2010]. Let $g : \mathbb{R}^n \to \mathbb{R}$. Then

$$\mathrm{Er}\, g(\bar{x}) \geq \liminf_{\substack{x \to \bar{x} \\ g(x) > g(\bar{x})}} \mathrm{dist}\,(0_n, \partial g(x)) = \mathrm{dist}\left(0_n, \operatorname*{Lim\,sup}_{\substack{x \to \bar{x} \\ g(x) > g(\bar{x})}} \partial g(x)\right).$$

$$\partial^{>} g(\bar{x}) = \operatorname*{Lim\,sup}_{\substack{x \to \bar{x} \\ g(x) > g(\bar{x})}} \partial g(x)$$

1. Regular locally Lipschitz function [Li, Meng, Yang, 2016]

$$\mathrm{dist}(0, \partial^{>} g(\bar{x})) \leq \mathrm{Er} g(\bar{x}) \leq \mathrm{dist}(0, \partial^{>}_{\sigma_{\partial g(\bar{x})}}(0))$$

Open question:

$$\partial^{>}_{\sigma_{\partial g(\bar{x})}}(0) \subset \partial^{>} g(\bar{x})?$$

Also an elegant piece of notation end $C$ was introduced, so

$$\partial^{>}_{\sigma_{\partial g(\bar{x})}}(0) = \mathrm{cl\,end}\,(\partial^{>}_{\sigma_{\partial g(\bar{x})}}(0)).$$

# Pointwise max of $C^1$ (Cánovas et al)

2. Let $g(x) = \max_{j \in J} g_j(x)$, where $g_j \in C^1$. Then

$$\bigcup_{D \in \mathcal{D}_{AI}(\bar{x})} \operatorname{co}\{\nabla g_j(\bar{x}), j \in D\} \subset \operatorname*{Lim\,sup}_{\substack{x \to \bar{x} \\ g(x) > g(\bar{x})}} \partial g(x),$$

where $\mathcal{D}_{AI}(\bar{x})$ contains all $D$ from $\mathcal{D}(\bar{x})$ such that $\{\nabla g_j(\bar{x}),\ j \in D\}$ are affinely independent.

Here $\mathcal{D}(\bar{x}) \subset 2^{J(\bar{x})}$, where $J(\bar{x})$ is the active index set, contains all subsets $D \subset J(\bar{x})$ such that the following system is consistent:

$$\begin{cases} \langle \nabla g_i(\bar{x}), d \rangle = 1, & j \in D, \\ \langle \nabla g_i(\bar{x}), d \rangle < 1, & j \in J(\bar{x}) \setminus D \end{cases}$$

in the variable $d \in \mathbb{R}^n$.

Open question: can we drop affine dependence?

# Pointwise max of $C^1$ (Cánovas et al)

3. Can we obtain some sort of characterisation for min-max functions, like the one for convex polyhedral functions [Cánovas, Henrion, López and Parra (2016)]

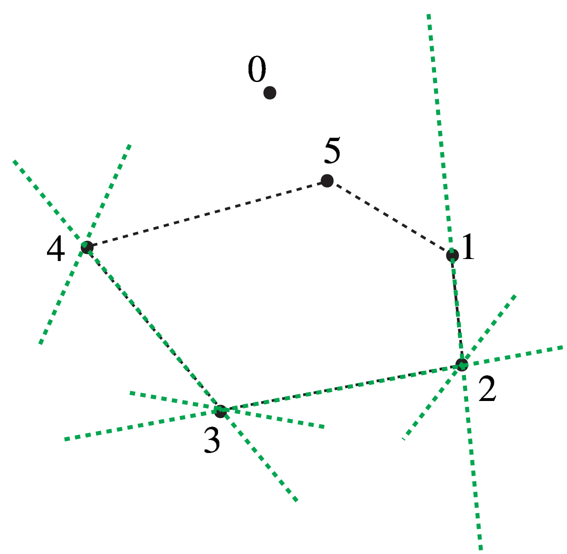
Let $g(x) = \max_{j\in J} g_j(x)$, where $g_j(x) = \langle a_j, x\rangle - b_j$, then

$$\operatorname*{Lim\,sup}_{\substack{x\to\bar{x}\\ g(x)>g(\bar{x})}} \partial g(x) = \bigcup_{D\in\mathcal{D}(\bar{x})} \operatorname{co}\{a_j, j\in D\},$$

where $\mathcal{D}(\bar{x}) \subset 2^{J(\bar{x})}$ consists of

all subsets $D \subset J(\bar{x})$ s.t. $\exists d$:

$$\begin{cases} \langle a_j, d\rangle = 1, & j\in D,\\ \langle a_j, d\rangle < 1, & j\in J(\bar{x})\setminus D \end{cases}$$



# Our contribution

**Theorem 1.** *Let* $f : \mathbb{R}^n \to \mathbb{R}$*,* $f(x) = \min_{i \in I} f_i(x)$*, where* $f_j(x)$ *is (Hadamard) directionally differentiable with sublinear directional derivative for all* $j \in J$*. Then*

$$\bigcup_{\substack{p \in \mathcal{S} \\ f'(\bar{x};p) > 0}} \bigcap_{i \in I(x,p)} \operatorname{Arg\,max}_{v \in \partial f_i(\bar{x})} \langle v, p \rangle \subset \operatorname*{Lim\,sup}_{\substack{x \to \bar{x} \\ f(x) > f(\bar{x})}} \partial f(x), \tag{1}$$

*where*

$$I(x,p) = \left\{ i_0 \in I(x) \,\middle|\, \max_{v \in \partial f_{i_0}(x)} \langle v, p \rangle = \min_{i \in I(x)} \max_{v \in \partial f_i(x)} \langle v, p \rangle \right\}.$$

# Question 1

**Corollary 2.** *Let* $f : X \to \mathbb{R}$ *be Hadamard directionally differentiable at every point* $x \in X$*, where* $X$ *is an open subset of* $\mathbb{R}^n$*; moreover assume that the directional derivative* $f'(\bar{x};\cdot)$ *is a sublinear function for every fixed* $\bar{x} \in X$*, then*

$$\bigcup_{\substack{p \in \mathcal{S} \\ f'(\bar{x};p)>0}} \operatorname*{Arg\,max}_{v \in \partial f(\bar{x})} \langle v, p \rangle \subset \operatorname*{Lim\,sup}_{\substack{x \to \bar{x} \\ f(x) > f(\bar{x})}} \partial f(x). \tag{2}$$

# Question 2

**Corollary 3.** *Let* $g(x) := \max_{j \in J} g_j(x)$*, with* $g_j : X \to \mathbb{R}$ *continuously differentiable for all* $j \in J$*, where* $|J| < \infty$*, and let* $\bar{x} \in \mathbb{R}^n$*. Then*

$$\bigcup_{D \in \mathcal{D}(\bar{x})} \operatorname{conv}\left\{\nabla g_j(\bar{x}),\ j \in D\right\} = \bigcup_{p \in \mathcal{S}, g'(\bar{x},p) > 0} \operatorname*{Arg\,max}_{v \in \partial g(\bar{x})} \langle v, p \rangle$$

$$\subseteq \operatorname*{Lim\,sup}_{\substack{x \to \bar{x} \\ g(x) > g(\bar{x})}} \partial g(x),$$

*in other words, the subsets* $D_{AI}(\bar{x})$ *can be replaced by* $D(\bar{x})$*.*

*Moreover when all* $\{g_j\}_{j \in J}$ *are affine we have an identity.*

# Question 3

**Theorem 4.** *Let* $f : X \to \mathbb{R}$ *be as before (i.e. finite min), and in addition assume that for every* $i \in I$ *the function* $f_i$ *is piecewise affine, i.e.*

$$f_i(x) = \max_{j \in J_i}(\langle a_{ij}, x\rangle + b_{ij}) \quad \forall i \in I,$$

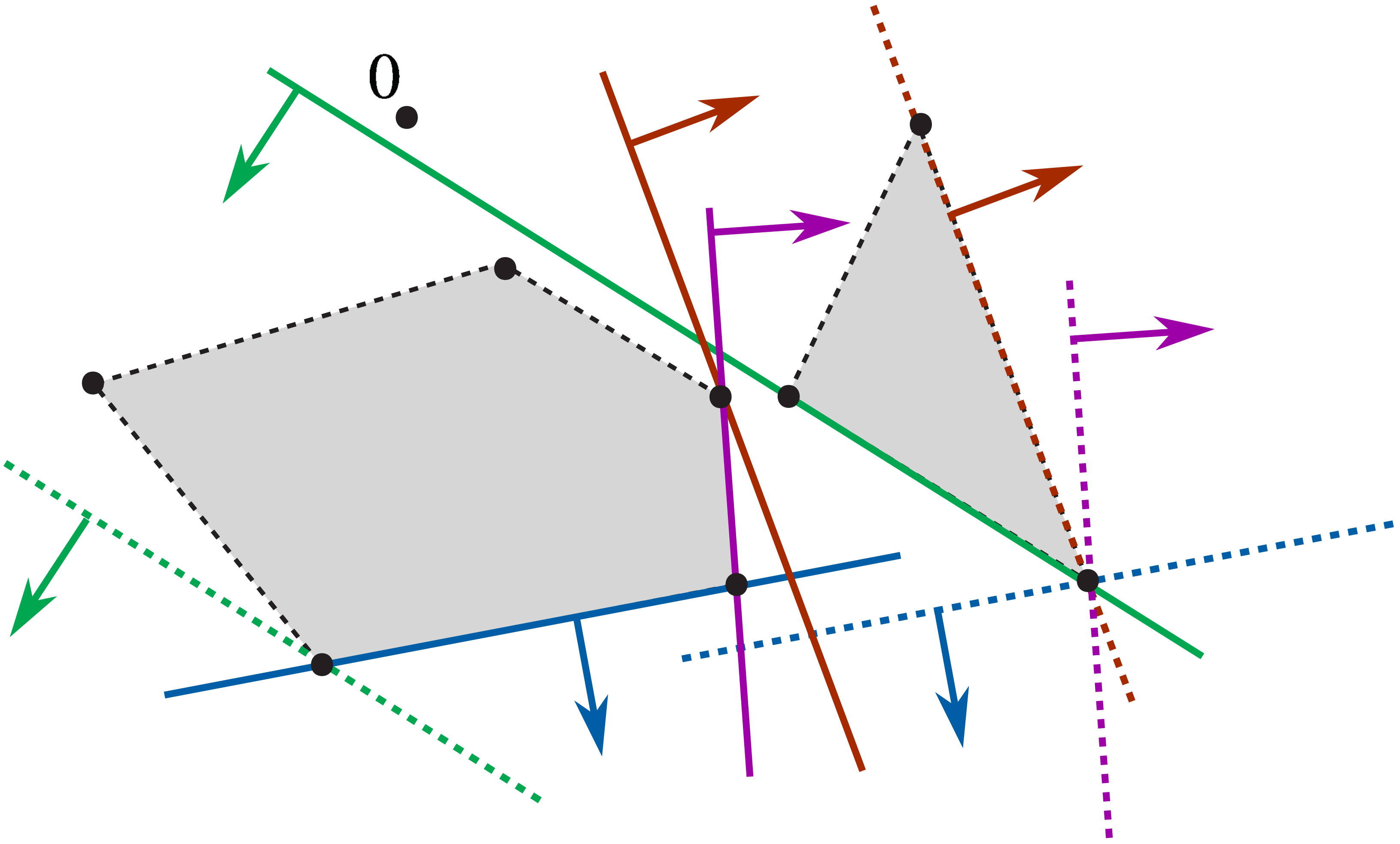
*where* $J_i$*'s are finite index sets for each* $i \in I$*. Then*

$$\bigcup_{\substack{p \in \mathcal{S} \\ f'(\bar{x};p)>0}} \bigcap_{i \in I(x,p)} \operatorname*{Arg\,max}_{v \in \partial f_i(\bar{x})} \langle v, p\rangle = \operatorname*{Lim\,sup}_{\substack{x \to \bar{x} \\ f(x) > f(\bar{x})}} \partial f(x),$$

*where as before*

$$I(x,p) = \left\{ i_0 \in I(x) \mid \max_{v \in \partial f_{i_0}(x)} \langle v, p\rangle = \min_{i \in I(x)} \max_{v \in \partial f_i(x)} \langle v, p\rangle \right\}.$$

# Geometric interpretation



# Main technical result

**Lemma 5.** *Let* $f : X \to \mathbb{R}$ *be a pointwise minimum of finitely many functions with sublinear Hadamard directional derivatives. Then for every* $\bar{x} \in X$, $p \in \mathcal{S}$ *and*

$$y \in \bigcap_{i \in I(\bar{x},p)} \operatorname{Arg\,max}_{v \in \partial f_i(\bar{x})} \langle v, p \rangle \tag{3}$$

*there exist sequences* $\{x_k\}$*and* $\{y_k\}$ *such that*

$$x_k \to \bar{x}, \quad \frac{x_k - \bar{x}}{\|x_k - \bar{x}\|} \underset{k\to\infty}{\longrightarrow} p, \quad y_k \in \partial f(x_k), \quad y_k \underset{k\to\infty}{\longrightarrow} y.$$

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