Outer limits of subdifferentials for min-max type functions

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Fréchet subdifferential

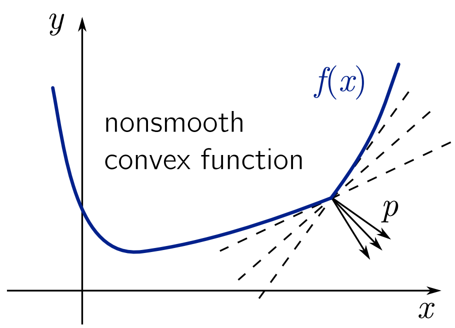
We will only consider continuous functions $f : \mathbb{R}^n \to \mathbb{R}$.

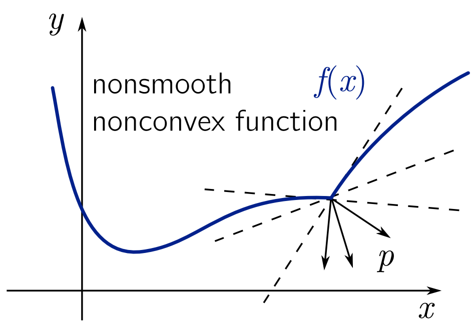
Fréchet subdifferential ∂f is a generalisation of the Moreau-Rockafellar subdifferential to nonconvex functions:

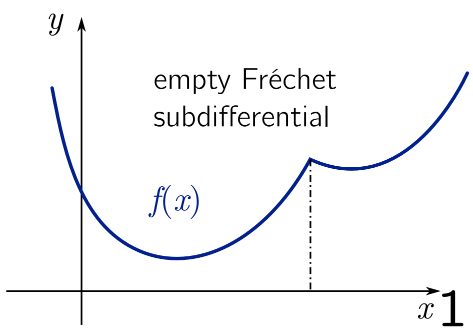
$$\partial f(x) = \left\{ v \,\middle|\, \liminf_{\substack{x' \to x \\ x' \neq x}} \frac{f(x') - f(x) - \langle v, x' - x \rangle}{\|x - x'\|} \geq 0 \right\}.$$

Compare with the subdifferential of a convex function:

$$\partial f(x) = \{ v \mid f(x') \geq f(x) + \langle v, x' - x \rangle \quad \forall x' \in \mathbb{R}^n \}.$$







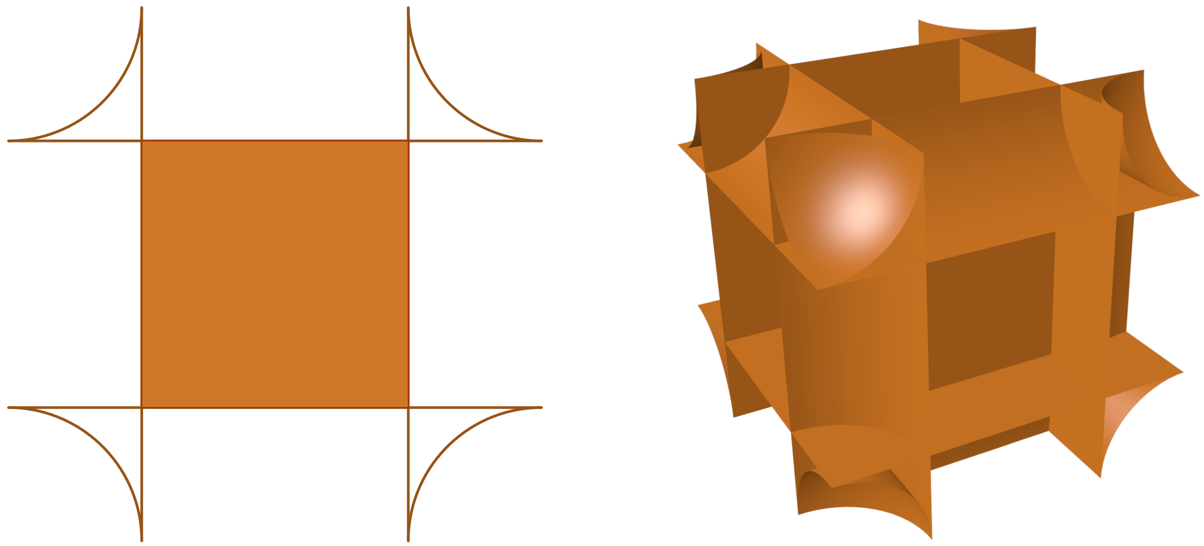
Limiting subdifferential

Limiting (Mordukhovich) subdifferential

$$\bar{\partial} f(\bar{x}) = \operatorname*{Lim\,sup}_{x \to \bar{x}} \partial f(x).$$

For example, consider the following function at $x = 0$.

$$f(x) = 2\|x\|_\infty - \|x\|_2$$



An old result

Let $\varphi_1, \varphi_2, \ldots, \varphi_r \colon \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz and approximate convex at $\bar{x}$. Then

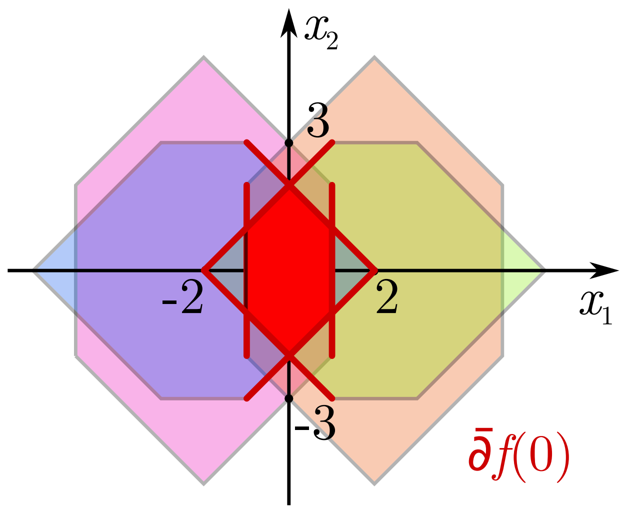
$$\bar{\partial}\left(\min_{i\in 1:r} \varphi_i\right)(\bar{x}) \subset \left(\bigcap_{i\in I(\bar{x})} \partial\varphi_i(\bar{x})\right) \cup \left(\bigcup_{g\in \mathcal{S}^{n-1}} C_{\bar{x}}(g)\right),$$

$$C_{\bar{x}}(g) = \left\{\bar{v} \,|\, \bar{v} \in \operatorname*{Arg\,max}_{\partial\varphi_{i_0}(\bar{x})}\langle v, g\rangle,\ \langle \bar{v}, g\rangle = \min_{i\in I(\bar{x})} \max_{v\in\partial\varphi_i(\bar{x})} \langle v, g\rangle\right\}$$

Approximate convexity is a technical assumption. Note that Daniilidis and Georgiev showed that a.c. locally Lipschitz functions coincide with the class of lower-C_1 functions in $\mathbb{R}^n$.

[Ngai, Luc, Thera (2000)], [Daniilidis, Georgiev (2004)], [R. (2010)]

Equality in the polyhedral case



[R. (2007)]

$$\bar{\partial}\left(\min_{i\in 1:r}\varphi_i\right)(\bar{x}) = \left(\bigcap_{i\in I(\bar{x})}\partial\varphi_i(\bar{x})\right)\cup\left(\bigcap_{g\in\mathcal{S}^{n-1}} C_{\bar{x}}(g)\right),$$

$$C_{\bar{x}}(g) = \left\{\bar{v} \mid \bar{v}\in \operatorname*{Arg\,max}_{\partial\varphi_{i_0}(\bar{x})}\langle v, g\rangle,\ \langle\bar{v}, g\rangle = \min_{i\in I(\bar{x})}\max_{v\in\partial\varphi_i(\bar{x})}\langle v, g\rangle\right\}$$

Motivation

Error bound bound [Fabian, Henrion, Kruger and Outrata, 2010]. For our purposes we assume that $g : \mathbb{R}^n \to \mathbb{R}$ is continuous. Then

$$\mathrm{Er}\, g(\bar{x}) \geq \liminf_{\substack{x \to \bar{x} \\ g(x) > g(\bar{x})}} \mathrm{dist}\,(0_n, \partial g(x)) = \mathrm{dist}\left(0_n, \operatorname*{Lim\,sup}_{\substack{x \to \bar{x} \\ g(x) > g(\bar{x})}} \partial g(x)\right).$$

$$\partial^{>} g(\bar{x}) = \operatorname*{Lim\,sup}_{\substack{x \to \bar{x} \\ g(x) > g(\bar{x})}} \partial g(x)$$

1. Regular locally Lipschitz function [Li, Meng, Yang, 2016]

$$\mathrm{dist}(0, \partial^{>} g(\bar{x})) \leq \mathrm{Er} g(\bar{x}) \leq \mathrm{dist}(0, \partial^{>}_{\sigma_{\partial g(\bar{x})}}(0))$$

Open question:

$$\partial^{>}_{\sigma_{\partial g(\bar{x})}}(0) \subset \partial^{>} g(\bar{x})?$$

The left hand side can be expressed via the end set,

$$\partial^{>}_{\sigma_{\partial g(\bar{x})}}(0) = \mathrm{cl}\,\mathrm{end}\,(\partial_{\sigma_{\partial g(\bar{x})}}(0)).$$

Min-max type functions

2. Can we obtain some sort of characterisation for min-max functions, like the one for convex polyhedral functions [Cánovas, Henrion, López and Parra (2016)]?

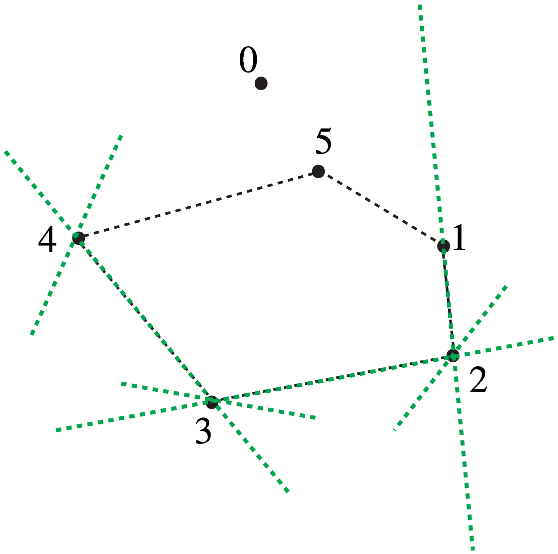
Let $g(x) = \max_{j \in J} g_j(x)$, where $g_j(x) = \langle a_j, x \rangle - b_j$, then

$$\operatorname*{Lim\,sup}_{\substack{x \to \bar{x} \\ g(x) > g(\bar{x})}} \partial g(x) = \bigcup_{D \in \mathcal{D}(\bar{x})} \mathrm{co}\{a_j, j \in D\},$$

where $\mathcal{D}(\bar{x}) \subset 2^{J(\bar{x})}$ consists of

all subsets $D \subset J(\bar{x})$ s.t. $\exists d$:

$$\begin{cases} \langle a_j, d \rangle = 1, & j \in D, \\ \langle a_j, d \rangle < 1, & j \in J(\bar{x}) \setminus D \end{cases}$$



Pointwise max of C^1

3. Let $g(x) = \max_{j \in J} g_j(x)$, where $g_j \in C^1$. Then

$$\bigcup_{D \in \mathcal{D}_{AI}(\bar{x})} \text{co}\{\nabla g_j(\bar{x}), j \in D\} \subset \operatorname*{Lim\,sup}_{\substack{x \to \bar{x} \\ g(x) > g(\bar{x})}} \partial g(x),$$

where $\mathcal{D}_{AI}(\bar{x})$ contains all D from $\mathcal{D}(\bar{x})$ such that $\{\nabla g_j(\bar{x}), j \in D\}$ are affinely independent.

Here $\mathcal{D}(\bar{x}) \subset 2^{J(\bar{x})}$, where $J(\bar{x})$ is the active index set, contains all subsets $D \subset J(\bar{x})$ such that the following system is consistent:

$$\begin{cases} \langle \nabla g_i(\bar{x}), d \rangle = 1, & j \in D, \\ \langle \nabla g_i(\bar{x}), d \rangle < 1, & j \in J(\bar{x}) \setminus D \end{cases}$$

in the variable $d \in \mathbb{R}^n$. [Cánovas, Henrion, López and Parra (2016)]

Open question: can we drop affine dependence?

Main result

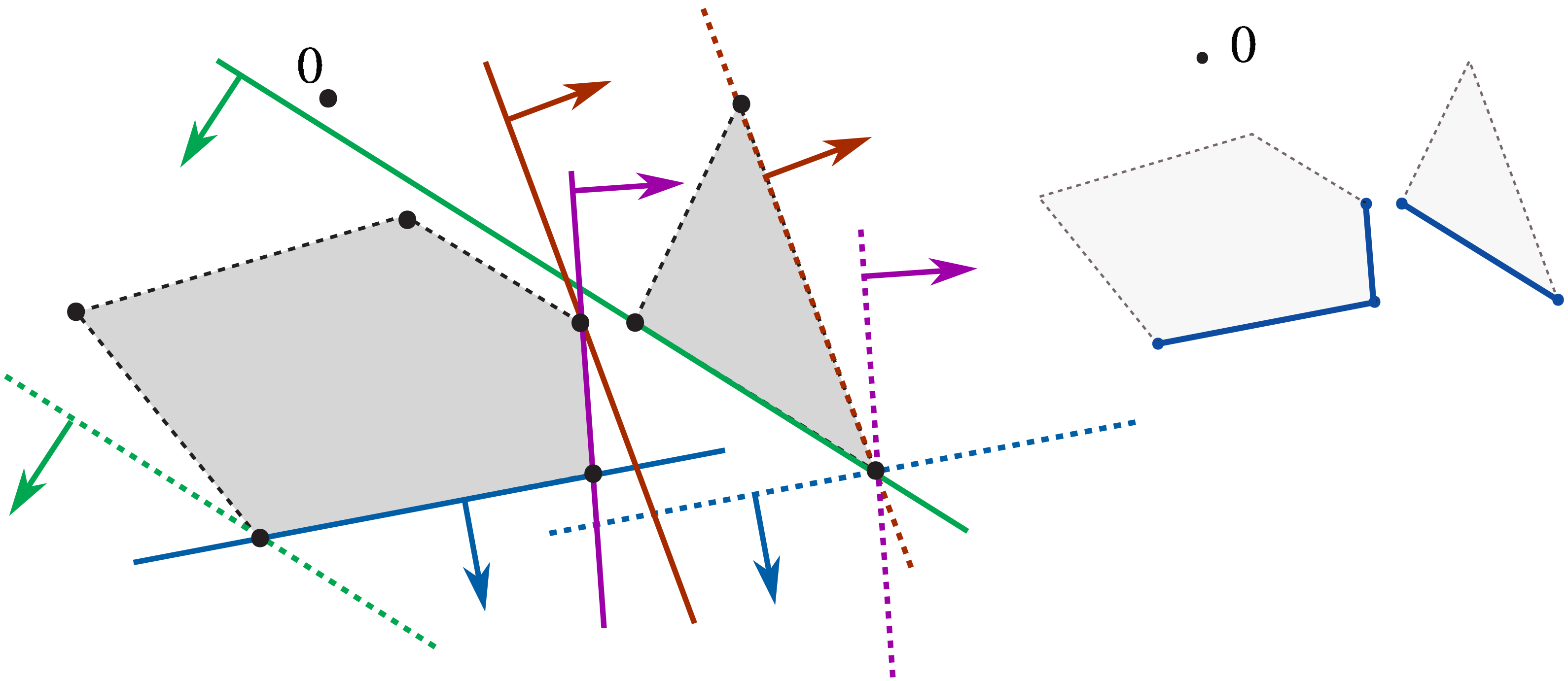
Theorem 1. *Let* $f : \mathbb{R}^n \to \mathbb{R}$*,* $f(x) = \min_{i \in I} f_i(x)$*, where* $f_i(x)$ *is (Hadamard) directionally differentiable with sublinear directional derivative for all* $i \in I$ *(think locally Lipschitz regular). Then*

$$\bigcup_{\substack{p \in \mathcal{S} \\ f'(\bar{x};p) > 0}} \bigcap_{i \in I(x,p)} \operatorname{Arg\,max}_{v \in \partial f_i(\bar{x})} \langle v, p \rangle \subset \operatorname*{Lim\,sup}_{\substack{x \to \bar{x} \\ f(x) > f(\bar{x})}} \partial f(x), \qquad (1)$$

where

$$I(x,p) = \left\{ i_0 \in I(x) \,\middle|\, \max_{v \in \partial f_{i_0}(x)} \langle v, p \rangle = \min_{i \in I(x)} \max_{v \in \partial f_i(x)} \langle v, p \rangle \right\}.$$

Geometric interpretation



Question 1

Corollary 2. *Let $f : X \to \mathbb{R}$ be Hadamard directionally differentiable at every point $x \in X$, where X is an open subset of $\mathbb{R}^n$; moreover assume that the directional derivative $f'(\bar{x}; \cdot)$ is a sublinear function for every fixed $\bar{x} \in X$, then*

$$\bigcup_{\substack{p \in \mathcal{S} \\ f'(\bar{x};p) > 0}} \operatorname*{Arg\,max}_{v \in \partial f(\bar{x})} \langle v, p \rangle \subset \operatorname*{Lim\,sup}_{\substack{x \to \bar{x} \\ f(x) > f(\bar{x})}} \partial f(x). \tag{2}$$

Question 2

Theorem 3. *Let $f : X \to \mathbb{R}$ be as before (i.e. finite min), and in addition assume that for every $i \in I$ the function f_i is piecewise affine, i.e.*

$$f_i(x) = \max_{j \in J_i}(\langle a_{ij}, x\rangle + b_{ij}) \quad \forall i \in I,$$

where J_i's are finite index sets for each $i \in I$. Then

$$\bigcup_{\substack{p \in \mathcal{S} \\ f'(\bar{x};p) > 0}} \bigcap_{i \in I(x,p)} \operatorname{Arg\,max}_{v \in \partial f_i(\bar{x})} \langle v, p\rangle = \operatorname*{Lim\,sup}_{\substack{x \to \bar{x} \\ f(x) > f(\bar{x})}} \partial f(x),$$

where as before

$$I(x,p) = \left\{ i_0 \in I(x) \mid \max_{v \in \partial f_{i_0}(x)} \langle v, p\rangle = \min_{i \in I(x)} \max_{v \in \partial f_i(x)} \langle v, p\rangle \right\}.$$

Question 3

Corollary 4. *Let* $g(x) := \max_{j \in J} g_j(x)$*, with* $g_j : X \to \mathbb{R}$ *continuously differentiable for all* $j \in J$*, where* $|J| < \infty$*, and let* $\bar{x} \in \mathbb{R}^n$*. Then*

$$\bigcup_{D \in \mathcal{D}(\bar{x})} \operatorname{conv}\left\{\nabla g_j(\bar{x}),\ j \in D\right\} = \bigcup_{p \in \mathcal{S}, g'(\bar{x}, p) > 0} \operatorname{Arg\,max}_{v \in \partial g(\bar{x})} \langle v, p \rangle$$

$$\subseteq \operatorname*{Lim\,sup}_{\substack{x \to \bar{x} \\ g(x) > g(\bar{x})}} \partial g(x),$$

in other words, the subsets $D_{AI}(\bar{x})$ *can be replaced by* $D(\bar{x})$*.*

Moreover when all $\{g_j\}_{j \in J}$ *are affine we have an identity.*

Main technical result

Lemma 5. *Let $f : X \to \mathbb{R}$ be a pointwise minimum of finitely many functions with sublinear Hadamard directional derivatives. Then for every $\bar{x} \in X$, $p \in \mathcal{S}$ and*

$$y \in \bigcap_{i \in I(\bar{x},p)} \operatorname*{Arg\,max}_{v \in \partial f_i(\bar{x})} \langle v, p \rangle \tag{3}$$

there exist sequences $\{x_k\}$and $\{y_k\}$ such that

$$x_k \to \bar{x}, \quad \frac{x_k - \bar{x}}{\|x_k - \bar{x}\|} \underset{k\to\infty}{\longrightarrow} p, \quad y_k \in \partial f(x_k), \quad y_k \underset{k\to\infty}{\longrightarrow} y.$$

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Thank you